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On Evaluation of Solutions to Equations of
Wave Propagation in a Layered Halfspace

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ON EVALUATION OF SOLUTIONS TO EQUATIONS OF WAVE
PROPAGATION IN A LAYERED HALFSpace

by

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ON EVALUATION OF SOLUTIONS TO EQUATIONS OF WAVE
PROPAGATION IN A LAYERED HALFSPACE

ABSTRACT

To remove certain discrepancies in solutions of problems of wave propagation in layered media, especially in the case of a two-layered liquid halfspace first discussed by Pekeris, the precise meaning of representations used is reconsidered. The existence of real poles on the path of integration used by Lamb and others for real ω leads to the addition of corresponding residues to the principal value of the integral. They represent the free waves usually superposed on the solutions to yield divergent waves. The question of real and complex poles is clarified for the Pekeris problem and the new form of solution is given in terms of a branch line integral and residues at all these poles. The Lamb condition of vanishing potential at infinity is fulfilled.

1. The solutions of problems on wave propagation in layered media by the well known methods of Lamb (1904) and Sommerfeld (1909, 1949) are usually obtained first in the form of certain integrals. These integrals can be improper and, therefore, use is to be made of their principal values in the Cauchy sense, as has been done by Lamb in the case of a homogeneous halfspace. It was pointed out by Kupradze and Sobolev (1930) that the procedure used by Lamb is equivalent to the choice of a definite value of a divergent integral. Further transformations of a solution of a wave problem lead to other difficulties of mathematical character which are not avoided in certain modifications of Lamb's method. For example, Pekeris (1948) derived expressions for potentials representing wave propagation in two-layered liquid halfspace in terms of an infinite number of normal modes and of a branch line integral. Recently Stone (1953) pointed out that Pekeris did not include in his solution an infinite number of complex poles. Since these poles are located in the fourth quadrant they yield in his solution terms exponentially increasing with depth. It is obvious that the existence of these terms presents a great difficulty to the theory and Stone derived another form of solution of the two-layer problem.

The existence of roots of the period equation, i. e. poles of integrands, was investigated by Sherman (1945) in more detail for the case of a liquid layer over a semi-infinite solid. He proved that there are an infinite number of simple complex roots of the period equation, besides a finite number of real roots, and pointed out that the point at infinity is an essen-

tial singularity of the integrands obtained in this problem.

The fact that there are certain discrepancies in representations of solutions, one of them being just mentioned, requires special attention. We have, therefore, to make a revision of the approach to certain points of the theory. The solution of the problem of wave propagation in a two-layered liquid halfspace will be used as an illustration.

2. We begin with some remarks concerning the expressions representing potentials due to a harmonic point source in a liquid layer overlying a semi-infinite liquid. We write the expressions obtained by Pekeris (1948) in the form

$$\varphi' = 2 \int_0^{\infty} K J_0(Kr) \frac{\sinh v'z}{v'} \frac{\cosh v'(H-h) + \sigma \frac{v''}{v'} \sinh v'(H-h)}{\Delta} dK \quad z < h \quad (1)$$

$$\varphi'' = 2 \int_0^{\infty} K J_0(Kr) \frac{\sinh v'h}{v'} \frac{\cosh v'(z-H) - \sigma \frac{v''}{v'} \sinh v'(z-H)}{\Delta} dK \quad z > h \quad (2)$$

$$\varphi_2 = 2\sigma \int_0^{\infty} K J_0(Kr) \frac{\sinh v'h}{v'\Delta} e^{-v''(z-H)} dK \quad (3)$$

where

$$\Delta = \cosh v'H + \sigma \frac{v''}{v'} \sinh v'H \quad (4)$$

H is the thickness of the layer and $\sigma = \frac{\rho_1}{\rho_2}$ is the ratio of densities. As usual r and z are cylindrical coordinates and k the wave number. In the

potential φ_2 for the semi-infinite liquid substratum ($H \leq z \leq \infty$) we have a factor $\exp[-\gamma''(z-H)]$. In order to obtain functions which will not diverge as $z \rightarrow \infty$ it is assumed that for a real $k > \frac{\omega}{\alpha_1}$ and $k > \frac{\omega}{\alpha_2}$ respectively

$$\gamma' = +\sqrt{k^2 - \frac{\omega^2}{\alpha_1^2}} \quad \gamma'' = +\sqrt{k^2 - \frac{\omega^2}{\alpha_2^2}} \quad (5)$$

are real and positive. In the problem we are considering the source is located at the point $(0, 0, h)$ of the layer.

The expressions (1-3) have characteristics typical of solutions of similar problems. The Bessel function $J_0(kr)$ in (1) is usually replaced by Hankel functions: $2J_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr)$ and in further transformations either $H_0^{(1)}$ or $H_0^{(2)}$ cancel out. The reason for the choice of one of these functions is immediately seen from their asymptotic form

$$H_0^{(1)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{i kr} \quad \text{provided } -\pi + 2\delta \leq \arg(kr) \leq 2\pi - 2\delta$$

$$H_0^{(2)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{-i kr} \quad \text{provided } -2\pi + 2\delta \leq \arg(kr) \leq \pi - 2\delta$$

δ being any positive acute angle. If one makes use of the time factor in the form $e^{i\omega t}$ the second Hankel function must be taken in the final expression.

To these preliminary remarks we add the following ones. Since the period equation

$$\Delta = \cosh \gamma' H + 5\gamma'' \frac{\sinh \gamma' H}{\gamma'} = 0 \quad (6)$$

yields in general a set of roots the expressions (1-3) can be improper integrals because of the existence of real poles of the integrands. Similar expressions are also improper integrals because of infinite limits. Therefore, we will use the notation $\mathcal{P}$ for their principal values in further transformations.

In (1-3) the summation is over the range of positive values of the real variable k in which, besides the poles, there are branch points of two radicals (5). We see, however, that the change in sign of γ' does not affect the expressions (1-3) and, therefore, the corresponding branch line integral vanishes as was found by Pekeris (l. c). A generalization of this important property of solutions for an n -layered halfspace was given by Jardetzky (1953).

According to Sherman's analysis mentioned above we can expect that the period equation (6) has a finite number of real roots in the interval $(\frac{\omega}{\alpha_2}, \frac{\omega}{\alpha_1})$. We usually consider this interval assuming that $\alpha_1 < \alpha_2$ since this is a more interesting case (a "fat bottom"), and well known simple considerations determine the number and location of real roots in it. As to the complex roots, Sherman's proof is based on the rule concerning the number of zeros and poles of a function of complex variable. We will not yet specify their number, mentioning first the conditions under which the evaluation of integrals of the type

$$\varphi = \int_0^{\infty} Q(\kappa) d\kappa \quad (7)$$

by contour integration are proved to be correct. The integrals in (1-3) have this form and the function Q is also a function of the set of variables:

$$r, z, \alpha_1, \alpha_2, h, H, \rho_1, \rho_2, \omega.$$

It is easy to see that the integrands in (1-3) can also be considered as functions of k^2 , i. e. can be replaced by an integral of the form

$$\varphi = \int_{-\infty}^{\infty} Q(\kappa) d\kappa \quad (8)$$

Now it has been proved that the principal value of such an integral can be computed by Cauchy theorem and is given by the formula (for k real)

$$\mathcal{P} \int_{-\infty}^{\infty} Q(\kappa) d\kappa = 2\pi i \left(\sum R + \frac{1}{2} \sum R_0 \right) \quad (9)$$

where $\sum R_0$ means the sum of residues of the poles on the real axis,

$\sum R$ is the sum of other residues, if the function $Q(k)$ satisfies certain conditions (Whittaker and Watson, 1947, p. 117). These conditions are as follows. Let us consider a complex variable $\zeta = \kappa + \tau i$ and a function $Q(\zeta)$ such that: 1) it is analytic when the imaginary part of ζ is positive or zero except at a finite number of poles; 2) it has a finite number of simple poles on the real axis; 3) as $|\zeta| \rightarrow \infty$, $\zeta Q(\zeta) \rightarrow 0$ uniformly for all values of $\arg \zeta$ such that $0 \leq \arg \zeta \leq \pi$ and 4) when k is real $kQ(k) \rightarrow 0$ as $k \rightarrow \pm \infty$ in such a way that $\int_0^{\infty} Q(\kappa) d\kappa$ and $\int_{-\infty}^0 Q(\kappa) d\kappa$ both converge. If there are trigonometric factors in Q the integrands can be subjected to less stringent conditions. The condition that $Q(\zeta)$ is a single-

valued function must be attended too (Whittaker and Watson l. c. p. 113).

These conditions are given in a form appropriate to the application of the Cauchy theorem in the upper halfplane. They can be easily modified when the integration occurs over the lower halfplane.

In this connection a remark has to be made concerning the use of the complex z plane. Because of the existence of branch points certain cuts must be used to determine the corresponding Riemann surface. Then, for example, the admissible first and fourth quadrants can belong to different sheets or to the same primary sheet for a multi-valued function. This fact can easily produce further confusion, if the integrand has singularities and if these points belong to different sheets of a Riemann surface. By the choice of a cut one can omit or introduce some singularities which are not appropriate to the solution.

It has been proved (Jardetzky, 1953) that in a problem of wave propagation in an n -layered medium all the integrands (like 1-3) are even functions of all γ_k (and γ_{ks} for shear waves) except for those exponents which refer to an infinite part of the medium (the last two, γ_n and γ_{ns} in case of a halfspace). Therefore all cuts but four referred to the branch points $\pm \frac{\omega}{\alpha_n}$ and $\pm \frac{\omega}{\beta_n}$ can be omitted. As to these four cuts they should be drawn always in such a way that no roots of the period equation which violate Lamb's condition are on the chosen part of the Riemann surface. This condition in a general case is that no negative real parts of the coefficients γ_n or γ_{ns} of a positive z in exponential factors $\exp(-\gamma z)$ of in-

tegrands are permitted. All these four branch points appear of course in the case of a homogeneous solid halfspace (Lapwood, 1949) and were considered also in the problem of a liquid layer over the solid halfspace (Sherman, 1945, compare Newlands, 1952).

3. Now writing the integral (1) or a similar one in the form

$$\varphi = 2 \int_0^{\infty} J_0(\kappa r) F(v', v'') \kappa d\kappa \quad (10)$$

we have to see the precise meaning given to it. There is an essential difference in problems considered by Lamb (1904) and Sommerfeld (1909). The latter, investigating the propagation of electromagnetic waves, obtained complex branch points and a complex pole so that in his solution there are no singular points of the integrand in (10) on the real axis. Thus no special definition of the integrals used was needed. A similar situation is found in application of Sommerfeld's method by Lapwood (1949) to elastic waves when a complex ω and operational analysis are used instead of a real frequency and Fourier analysis. For real ω , however, beginning with Lamb's problem we find in problems concerning wave propagation in layered media singularities on the real k axis. A precise definition of expression (10) is, therefore, required and this can be obtained from the discussion given by Whittaker and Watson (l. c.).

The integration in (10) has to be performed along the positive part of the real axis between the poles and along infinitely small half-

circles γ_n with centers at these points. Thus if x_n are the poles

$$\varphi = \lim \left\{ \int_0^{x_1 - \tau_1} + \int_{\gamma_1} + \int_{x_1 + \tau_1}^{x_2 - \tau_2} + \int_{\gamma_2} + \dots \right\}$$

provided the radii τ_n tend to zero. Now the sum of odd terms is the principal value of the integral. The terms taken along the half-circles

γ_n yield the residues corresponding to real poles which are obtained by the transformation $z = x_1 + \tau_1 e^{i\theta}$

$$\int_{\gamma_1} Q(z) dz = i \int_{-\pi}^0 Q(x_1 + \tau_1 e^{i\theta}) \tau_1 e^{i\theta} d\theta$$

and $\lim_{z \rightarrow x_1} [Q(z)(z - x_1)] = \text{Res.}$ Thus

$$\varphi = \mathcal{P} \int_0^\infty 2J_0(\kappa r) F(v', v'') \kappa d\kappa - \pi i \sum 2R_0 \quad (11)$$

where the sum is taken over a finite number of real poles. Equation (11) yields the explanation of the fact that in order to obtain a train of waves diverging from the source Lamb had to add "free Rayleigh waves" to that part of his solution which is represented by principal parts of integrals. This train of "free" waves is exactly given by the last terms in (11).

An attempt has been made by Lapwood to explain the addition of free waves as due to the special choice of the Riemann surface made by Lamb. It is obvious that choosing the path of integration as a real axis under other conditions we will obtain different definitions of the integral (10). Especially, using different rules for construction of a Riemann surface, we will have to vary the contours. However, the relationship

(11) is due to the existence of real roots of the period equation and holds for a single-valued function as well as for a multivalued integrand for which the use of a Riemann surface is necessary for contour integration. The superposition of free waves appears now as a natural part of the precise definition of an integral representing the solution of a problem of wave propagation and loses the character of an arbitrary transformation.

4. The second point in the definition of (10) which requires more precision is to obtain a uniform function as an integrand, as mentioned before. In the case of the integral (10) we have a two-leaved Riemann surface because of two values of the radical $\sqrt{}$. It is important to indicate all conditions under which the choice of such a surface is made, especially those concerning an appropriate cut. Sherman and others, for example, used as a cut the part of the real axis between the branch points $\pm K_{\alpha_2} = \pm \frac{\omega}{\alpha_2}$ (see equation 3). Pekeris made use of rays starting at the branch points k_{α_1} and k_{α_2} and parallel to the negative imaginary axis. Such cuts seem to produce no confusion in Lamb's problem but they were also used in many other investigations. This choice, according to Stone (1953), is responsible for inclusion of terms having no physical meaning. In discussions of wave propagation in different cases either upper or lower and also the left-hand or right-hand halfplanes were used as a part of the admissible sheet of a Riemann surface. We will take the latter. To get more insight into all these questions it is easier

to begin the discussion for complex ω as it was done by Lapwood.

Since in potentials for the infinite part of a medium we have a factor $\exp(-\gamma''z)$ $\text{Re } \gamma''$ cannot be negative for positive z as has been assumed above. Now putting

$$\gamma'' = a + bi, \quad \zeta = \kappa + \tau i, \quad \omega = q - ip \quad (12)$$

we can find the curve determined by the condition

$$\text{Re } \gamma'' \equiv a = 0 \quad (13)$$

By (5)

$$-b^2 = \kappa^2 - \tau^2 + 2\kappa\tau i - \frac{q^2 - p^2 - 2qpi}{\alpha_2^2} \quad (14)$$

Thus the line in question is a part of the hyperbola (Fig. 1a).

$$\kappa\tau = -\frac{qp}{\alpha_2^2} \quad (15)$$

It starts at the branch point $\zeta = \kappa_{a_2} = \frac{\omega}{\alpha_2}$ in the fourth quadrant, if we restrict ourselves now by the case $q > 0$, $p > 0$. This follows from the second condition given by (12), namely,

$$\kappa^2 - \tau^2 < \frac{q^2 - p^2}{\alpha_2^2} \quad (16)$$

We can easily see that the second part of the hyperbola (15), that on the other side of the branch point, corresponds to the conditions

$$\text{Im } \gamma'' \equiv b = 0 \quad (17)$$

and

$$\kappa^2 - \tau^2 > \frac{q^2 - p^2}{\alpha_2^2} \quad (18)$$

At the branch point k_{α_2} the real and imaginary parts of γ'' both vanish and a similar distribution around the branch point $-k_{\alpha_2}$ occurs on the second branch of the hyperbola (15) in the second quadrant. A change in sign of p or q will displace these curves into first and third quadrant respectively.

There are two extreme cases: $\omega = q$ is real and $\omega = -ip$ is purely imaginary, in which (15) becomes

$$\kappa\tau = 0 \quad (19)$$

For ω real we obtain by (15 - 16): 1) $\tau = 0$, $k^2 < \frac{q^2}{\alpha_2^2}$,
i. e. a part of the real axis between the branch points, or 2) $k=0$, $-\tau^2 < \frac{q^2}{\alpha_2^2}$
i. e. the imaginary axis. Taken separately this axis does not represent a cut since it does not pass through a branch point. It can be, however, combined with the line BOA (Fig. 1b) to form two cuts AOD and BOC which obviously represent the degenerated branches of the hyperbola (15) if $p \rightarrow 0$ (Lapwood, 1949).

If we take $\omega = -ip$, $p > 0$, i. e. the time factor $e^{i\omega t} = e^{pt}$ used first by Cagniard (1939) we obtain: 1) $k=0$, $\tau^2 > p^2$ i. e. cuts along the imaginary axis or 2) $\tau = 0$, $k^2 < -p^2$, i. e. conditions which cannot be satisfied.

We will now consider the case of ω real in more detail. The lines AOD and BOC are determined by the condition (13) and $\text{Re } \gamma''$ can either change sign or remain positive or negative on both sides. Now for a positive z we have to associate a positive $\text{Re } \gamma''$ with points to the right

of the point A on the real axis. This first condition will determine the admissible sheet of the Riemann surface for this case. Now it can be shown that under appropriate conditions the right-hand halfplane of z belongs to it. Let us put

$$z = \kappa_{\alpha 2} + \tau_{\alpha} e^{i\theta} \quad (20)$$

Then by (3)

$$\gamma''^2 = 2\kappa_{\alpha 2} \tau_{\alpha} e^{i\theta} + \tau_{\alpha}^2 e^{2i\theta} \quad (21)$$

and in the neighborhood of A on neglecting τ_{α}^2 we obtain

$$\gamma''^2 = 2\kappa_{\alpha 2} \tau_{\alpha} e^{i\theta}$$

Hence

$$\begin{aligned} \gamma_1'' &= +\sqrt{z^2 - \kappa_{\alpha 2}^2} = \sqrt{2\kappa_{\alpha 2} \tau_{\alpha}} e^{i\frac{\theta}{2}} \\ \gamma_2'' &= -\sqrt{z^2 - \kappa_{\alpha 2}^2} = \sqrt{2\kappa_{\alpha 2} \tau_{\alpha}} e^{i(\frac{\theta}{2} + \pi)} \end{aligned} \quad (22)$$

If we assume that in the right-hand halfplane the principal value of the angle θ is used, i. e. $-\pi < \theta \leq \pi$ it is the first definition (22) of the radical which yields a positive $\text{Re } \gamma''$ in this halfplane. Obviously $\text{Im } \gamma''$ is then positive in the first quadrant and negative in the fourth. To insure the vanishing of $\exp(-\gamma'' z)$ at $z \rightarrow \infty$ for $z < 0$ we should use another convention of signs (e. g. the second definition (22)) or change the sign in the exponent. In case of a complex ω $\text{Im } \gamma''$ is positive on one side of the hyperbola and negative on the other (Fig. 1a). On a part of the hyperbola or on the part Ak of the real axis (Fig. 1b) where $\text{Im } \gamma'' = 0$, γ'' is continuous. It is discontinuous along OA.

If BOC or BOA is the cut we put for the neighborhood of point B

$$\zeta = -\kappa_{\alpha 2} + \tau_{\alpha} e^{i\theta}$$

and vary the angle θ in the interval $0 \leq \theta < 2\pi$ in order not to cross the cut. Then

$$\begin{aligned} \gamma''^2 &= -2\kappa_{\alpha 2}\tau_{\alpha}e^{i\theta} + \tau_{\alpha}^2e^{2i\theta} \\ \gamma_1'' &= \sqrt{2\kappa_{\alpha 2}\tau_{\alpha}}e^{i\frac{\theta+\pi}{2}}, \quad \gamma_2'' = \sqrt{2\kappa_{\alpha 2}\tau_{\alpha}}e^{i(\frac{\theta}{2} + \frac{3\pi}{2})} \end{aligned} \quad (22')$$

$\text{Re } \gamma_1''$ is now negative if the first definition is taken.

For the further transformation of the integral (10) we need only the right-hand halfplane and it is important to note that under the condition that the angle θ has its principal value the definition of γ'' along OA is not changed. The change in sign of the radical occurs only for its imaginary part, the $\text{Re } \gamma''$ being zero at every point on OA and positive on both sides of it. At A $\text{Re } \gamma'' = 0$ and $\text{Im } \gamma'' = 0$.

Now using this definition of γ'' the integrand in (10) becomes in the right-hand halfplane a single-valued function of ζ as is required for the contour integration. If the integrand is a continuous function of ζ it is also a continuous function of $\text{Re } \gamma''$ but a discontinuous one of $\text{Im } \gamma''$ along the cut since on different sides of it the radical which becomes equal to its imaginary part has opposite signs. When real ω is used the part OA of the cut coincides with a part of the integration path. To give a precise meaning to the integral we have to take, for example, the

value of γ'' on the upper side of the cut. Given by Lamb as a part of a convention on signs of radicals this seems to be arbitrary because such a formal choice of signs does not show the true reason for it. The existence of the discontinuity along the cut explains now the appearance of the branch line integral.

If we put $\gamma'' = a + bi$ the real and imaginary parts can be computed for every given point z from (22). Then, since a^2 and $-b^2$ are roots of a certain quadratic equation, we obtain, for example, $a = \pm \sqrt{u_1}$. The convention to be attended is that $\text{Re } \gamma'' > 0$ in the first and fourth quadrant, i. e. $a = \sqrt{u_1}$.

5. We mentioned the fact that in case of a liquid over a solid halfspace there are an infinite number of complex roots of the period equation. Since this case includes the problem of a liquid halfspace we can assume that Sherman's proof also shows the existence of such roots in our problem. To locate them we can use a simple transformation as follows:

Writing exponential functions in (6) we obtain the period equation in the form

$$\frac{1}{2} \left[\left(1 + \frac{\sigma \gamma''}{\gamma'} \right) e^{2\gamma'' H} + \left(1 - \frac{\sigma \gamma''}{\gamma'} \right) \right] e^{-\gamma' H} = 0 \quad (23)$$

or

$$e^{2\gamma'' H} = \frac{\sigma \gamma'' - \gamma'}{\sigma \gamma'' + \gamma'} \quad (24)$$

This is equivalent to

$$2\gamma'H - \ln \frac{6\gamma'' - \gamma'}{6\gamma'' + \gamma'} - n \cdot 2\pi i = 0 \quad (25)$$

Every root of (25) satisfies (23) and vice versa. Substituting (5) and a complex z for k we obtain

$$2H \sqrt{z^2 - \kappa_{\alpha_1}^2} - \ln \frac{6\sqrt{z^2 - \kappa_{\alpha_2}^2} - \sqrt{z^2 - \kappa_{\alpha_1}^2}}{6\sqrt{z^2 - \kappa_{\alpha_2}^2} + \sqrt{z^2 - \kappa_{\alpha_1}^2}} - n \cdot 2\pi i = 0 \quad (25')$$

We will assume now that $|z| \gg 1$ and put $z_* = \pm z$, writing the lefthand member of (25') in the form

$$G(z_*) = 2z_* H \left(1 - \frac{\kappa_{\alpha_1}^2}{z_*^2}\right)^{\frac{1}{2}} - \ln \frac{6\left(1 - \frac{\kappa_{\alpha_2}^2}{z_*^2}\right)^{\frac{1}{2}} - \left(1 - \frac{\kappa_{\alpha_1}^2}{z_*^2}\right)^{\frac{1}{2}}}{6\left(1 - \frac{\kappa_{\alpha_2}^2}{z_*^2}\right)^{\frac{1}{2}} + \left(1 - \frac{\kappa_{\alpha_1}^2}{z_*^2}\right)^{\frac{1}{2}}} - n \cdot 2\pi i \quad (26)$$

Expanding this expression for large values of $|z|$ we can restrict ourselves by considering the terms

$$2z_* H \left(1 - \frac{\kappa_{\alpha_1}^2}{2z_*^2}\right) + \frac{2}{6} \left[1 - \frac{1}{z_*^2} (\kappa_{\alpha_1}^2 - \kappa_{\alpha_2}^2)\right] - n \cdot 2\pi i \quad (27)$$

An even more rough approximation leads us to the conclusion that the roots are in the fourth and second quadrant respectively. If we put, namely,

$$G(z_*) \approx 2Hz_* + \frac{2}{6} - n \cdot 2\pi i = 0 \quad (28)$$

we see that there is a root

$$\pm z = z_* \approx -\frac{1}{6H} + \frac{\pi n}{H} i \quad (29)$$

provided n is large enough to make the modulus of ζ a large number.

This was the condition under which the approximation has been made.

Since the definition (5) has been used and it holds for the fourth quadrant of the admissible sheet of the Riemann surface, there are complex roots on it. It should be noted that there are no roots of the period equation (6) on the imaginary axis. If we put $\zeta = -im$ this equation takes the form

$$\cos \alpha_1 H + i \delta \frac{a_2}{a_1} \sin \alpha_1 H = 0$$

where $a_1 = \sqrt{m^2 + \kappa_{\alpha_1}^2}$, $a_2 = \sqrt{m^2 + \kappa_{\alpha_2}^2}$. It has no real roots m .

The change in sign of γ'' , i. e. the use of the second definition of the radical (22) will affect the form of the period equation (6). Since the sign of the second term of this equation will be reversed the term $\ln$ in (25) will be positive. Then it may be easily seen that the roots of the new period equation are located in the first and third quadrant respectively.

6. The discussion of the integrand in (10) has shown that there are three basic characteristics in case of ω real: 1) the finite number of real roots in the interval $(\kappa_{\alpha_2}, \kappa_{\alpha_1})$; 2) infinite number of complex roots in the fourth quadrant; 3) the part of the integration path from 0 to κ_{α_2} is a part of the branch line. On its two sides γ'' has purely imaginary values with opposite signs. To give a precise meaning of the integral (10) we must use one of them, say the positive one in (10) and (11).

If we now substitute $2J_0(\kappa r) = H_0^{(1)}(\kappa r) + H_0^{(2)}(\kappa r)$ in the principal value we obtain

$$I = \mathcal{P} \int_0^\infty [H_0^{(1)}(\kappa r) + H_0^{(2)}(\kappa r)] F(\gamma', \gamma'') \kappa d\kappa = I_1 + I_2 \quad (30)$$

In the first integral we can use the contour enclosing the first quadrant where there are no other roots than those on the real axis (Fig. 2). The Cauchy theorem (see equation 9) yields

$$I_1 = \pi i \sum_0^+ + \int_0^{i\infty} H_0^{(1)}(\zeta r) F(\nu', \nu_+'') \zeta d\zeta \quad (31)$$

where R_0^+ are residues at real poles when one of the factors is the first Hankel function, $\zeta = \tau i$, and the subscript $+$ shows that along the positive imaginary axis we have $\nu'' = \text{Im } \nu'' > 0$.

To transform the integral I_2 in a similar way we must take into account the fact that below the cut $\text{Im } \nu''$ is negative. Therefore we will add and subtract $\int_0^{\kappa_{d2}} H_0^{(2)}(\kappa r) F(\nu', \nu_-'') \kappa d\kappa$,

write

$$I_2 = I' + \int_0^{\kappa_{d2}} H_0^{(2)}(\kappa r) [F(\nu', \nu_+'') - F(\nu', \nu_-'')] \kappa d\kappa \quad (32)$$

and apply the countour integration to the sum I' . Then

$$I' = \int_0^{-i\infty} H_0^{(2)}(\zeta r) F(\nu', \nu_-'') \zeta d\zeta - 2\pi i \sum R - \pi i \sum R_0^- \quad (33)$$

where $\sum R_0^-$ is the sum of residues for real roots and the function $H_0^{(2)}$ is one of the factors. R is a residue for one of the complex poles (ζ_n) in the fourth quadrant. Taking into account the relationship

$$H_0^{(1)}(Z) = -H_0^{(2)}(-Z)$$

the last term in (31) is ($\zeta = \tau i$)

$$\int_0^\infty H_0^{(2)}(-\tau r i) F(\nu', \nu_+'') \tau d\tau = \int_0^{-\infty} H_0^{(2)}(\tau r i) F(\nu', \nu_+'') \tau d\tau \quad (34)$$

Now adding (31), (34) we obtain

$$I = \pi i (\sum R_0^+ - \sum R_0^-) - 2\pi i \sum R + \int_0^{\kappa_{d2}} H_0^{(2)} [F(v', v_+'') - F(v', v_-'')] \kappa d\kappa + \int_0^{-\infty} H_0^{(2)}(\tau r i) [F(v', v_+'') - F(v', v_-'')] \tau d\tau \quad (35)$$

Since in (11) the residues are computed for the factor $2J_0$ we have

$$2\sum R_0 = \sum R_0^+ + \sum R_0^- \quad (36)$$

the first sum in the right hand member being taken for the Hankel function $H_0^{(1)}$, the second for $H_0^{(2)}$.

Then by (8), (9), (35) and (36)

$$\varphi = 2 \int_0^{\infty} J_0(\kappa r) F(v', v'') \kappa d\kappa = -2\pi i \sum R_0^- - 2\pi i \sum R + \int_{-\infty i}^0 H_0^{(2)}(z r) [F(v', v_+'') - F(v', v_-'')] z dz + \int_0^{\kappa_{d2}} H_0^{(2)}(\kappa r) [F(v', v_+'') - F(v', v_-'')] \kappa d\kappa \quad (37)$$

The residues R_0^- and R have to be computed for the integrand

$H_0^{(2)}(z r) F(v', v_-'') z$ which for the real poles takes the form $H_0^{(2)}(\kappa r) F(v', v_-'') \kappa$

and $v_-'' = \text{Re } v''$. The sum of the last two terms is the branch line integral taken along the cut $-i\infty, 0A$.

It represents superposition of a

continuous wave spectrum. It is well known that when such integrals are

evaluated by different methods they are approximated by a certain number

of waves in a different way. As to the function F its form is seen in

three expressions (1-3) which hold for wave propagation in a two-layered

liquid halfspace. The integrands thus have the form $H_0^{(2)}(zr) M_2(z) / N_2(z)$.

According to the well known formula the residue at a simple pole (k_n) is

$$R_{0n}^- = H_0^{(2)}(k_n r) M_2(k_n) / N_2'(k_n)$$

In (1-3) $N_2 \equiv \Delta$ and N_2' is its derivative with respect to z ,

k_n being the real roots of the period equation $N_2(k_n) = 0$.

The solution of this problem, represented by three equations of the type (37), differs from those given in investigations of Pekeris and Stone quoted before. As mentioned above, Stone pointed out that had Pekeris introduced the second sum of residues in (37) which correspond to complex roots, they should be associated with a negative $\text{Re } \gamma''$ because of his choice of the cut, and therefore would violate the condition at infinity ($z \rightarrow \infty$). Besides these terms there are in (37) only a finite number of residues R_0 corresponding to real roots. The branch line ~~integral~~ in (37) differs also from the straight line from $k_{\alpha 2}$ to $k_{\alpha 2} - i\infty$ used by Pekeris, and also from that given by Stone. The latter obtained in his final expression the integral from 0 to $k_{\alpha 2}$ only since he assumed that Pekeris' proof of the fact that terms like the last in (31) and the first in (33) cancel out holds also for his new choice of the cuts. For the cuts shown in Figure 1b, however, the values of $\gamma'' = \text{Im } \gamma''$ on the imaginary axis differ in sign on its positive and negative parts as we have seen before and indicated by the subscripts + and -. Comparing the distribution of $\text{Re } \gamma''$ and $\text{Im } \gamma''$ given in Figures 1a and 2, it may be easily seen that the Pekeris

conclusion about the vanishing of integrals is correct since it corresponds to the case 1a. It does not hold for case 2 considered by Stone. Therefore in our case the part of the branch line integral from $-i\infty$ to 0 does not vanish. In Stone's solution where a finite number of normal modes is obtained, the sum of residues such as the second in (37) is omitted, but using Sommerfeld's cuts we have a positive $\text{Re } \gamma''$ in the whole right-hand halfplane which is associated with the first definition (22) of the radical. We have seen that this halfplane belongs to the admissible sheet of the Riemann surface. The roots on it do not violate the condition at infinity.

The new form of solution of the problem, i. e. the equations (37) representing potentials of wave propagation in a two-layered liquid half-space, obviously does not affect the conclusions for the field involving phase and group velocities reached by Pekeris and others. These applications of the theory are based on consideration of the period equation.

We discussed here only the case of one radical which makes the integrands double valued functions. A generalization for multi-valued functions requires no essential changes.

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